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STABILITY OF A FORCED CONVECTION MASS TRANSFER SYSTEM WITH A CONCENTRATION-DEPENDENT DIFFUSIVITY

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NOMENCLATURE

c,	concentration of solute;
$\bar{c}$,	concentration of solute at the interface;
c_{St},	concentration of solute corresponding to the stationary state;
D,	diffusion coefficient;
f,	function defined by equation (19);
g,	function defined by equation (14);
k,	constant defined by equations (10) and (11);
q,	function defined by equation (15);
t,	time;
v_0,	maximum velocity;
x, z,	direction coordinates.

Greek symbols

δ,	liquid film thickness;
η,	parameter defined by equation (2);
ϕ,	parameter defined by equation (3).

INTRODUCTION

IT IS well-established that the model of a stable interface between two fluid phases does not in general hold when heat and mass transfer takes place. These transfer processes cause interfacial concentration and temperature gradients which in turn are accompanied by surface tension and buoyancy differences which induce interfacial flow. Numerous studies, both theoretical and experimental, have been made regarding the nature and effect of these interfacial phenomena which have been grouped together under the name 'Marangoni effects', and these investigations have been critically reviewed by Berg [1].

In this department, when a series of measurements were made to determine gas–liquid diffusivities, it was observed for systems where Marangoni instabilities were known to arise that a minimum value of diffusivity was always exhibited at some concentration. A recent investigation by Pratt and Wakeham [2] when measuring the mutual diffusion coefficient of ethanol–water mixtures demonstrates a minimum diffusivity value at approximately 0.27 ethanol mole fraction, the minimum value being about one third of the value for pure water at the same temperature. Similar behavior is reported for the system acetone–water by Tyn and Calus [3] and for a number of other binary mixtures given in a very detailed review by Dullien *et al.* [4].

The purpose of this note is to ascertain if the hypothesis proposed has theoretical justification by examining the development of a linear stability criterion for the case of a non-linear concentration-dependent diffusion process.

ANALYSIS

Physical model: diffusion into a falling film in Cartesian coordinate system

The problem under consideration is a case of forced-convection mass transfer in which viscous flow and diffusion occur. It is considered that the absorption of the gas takes place in a laminar liquid film. The viscosity of the solution is not changed significantly by the absorption process. Furthermore, it is assumed that the diffusion process is so slow in the liquid film that the solute penetration distance in the film is small in comparison with the film thickness. The system is illustrated in Fig. 1.

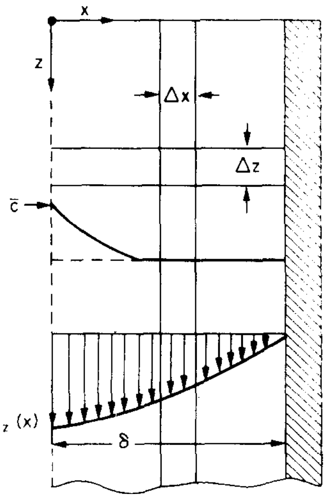

FIG. 1. Schematic of physical problem.

The differential equation describing the process is (Bird *et al.* [5]):

$$\frac{\partial c}{\partial t}+v_z(x)\frac{\partial c}{\partial z}=\frac{\partial}{\partial x}\left[D\frac{\partial c}{\partial x}\right] \tag{1}$$

where

$$v_z(x)=v_0\left[1-\left(\frac{x}{\delta}\right)^2\right].$$

For short contact times one can consider that the film is infinitely thick and moves with velocity v_0. Using the following transformations the equation (1) can be reduced to a two-parameter differential equation

$$\eta = x/2\sqrt{z} \tag{2}$$

$$\phi = t/z. \tag{3}$$

Thus,

$$4\frac{\partial c}{\partial \phi}\left[1-\phi v_z(x)\right]-2\eta v_z(x)\frac{\partial c}{\partial \eta}=\frac{\partial}{\partial \eta}\left[D\frac{\partial c}{\partial \eta}\right]. \tag{4}$$

Equation (4) is not separable since $D = D[c(\eta, \phi)]$ and v_z is x-dependent. By assuming a short contact time constraint, $v_z(x) = v_0$ = a constant for a given system; or $x/\delta \ll 1$.

With this assumption equation (4) reduces to

$$4\frac{\partial c}{\partial \phi}(1-\phi v_0)-2\eta v_0\frac{\partial c}{\partial \eta}=\frac{\partial}{\partial \eta}\left[D\frac{\partial c}{\partial \eta}\right]. \tag{5}$$

The stability equations

Let c_{St} be the stationary solution of the time-independent equation

$$-2\eta v_0\frac{\partial c}{\partial \eta}=\frac{\partial}{\partial \eta}\left[D\frac{\partial c}{\partial \eta}\right]. \tag{6}$$

It is evident from equation (6) that c is only dependent on η and satisfied the boundary conditions:

$$c(0)=\bar{c};\quad c(\infty)=0.$$

The problem now is to examine the stability criteria of the diffusion equation subjected to small perturbations, $\Phi(\phi,\eta)$ about the stationary state.

The perturbed concentration profile is thus

$$c(\phi,\eta)=c_{St}(\eta)+\Phi(\phi,\eta). \tag{7}$$

Substitution of equation (7) in equation (5) with the approximation that since the perturbation is small, only linear terms in Φ, $\partial\Phi/\partial\phi$, $\partial\Phi/\partial\eta$ and $\partial^2\Phi/\partial\eta^2$ exist. For $\partial c_{St}/\partial\phi=0$, one can obtain the following expression:

$$\begin{aligned}&4\frac{\partial\Phi}{\partial\phi}(1-\phi v_0)\\&\quad=D(c_{St})\frac{\partial^2\Phi}{\partial\eta^2}+\left[2\eta v_0+\frac{2\,\mathrm{d}D(c_{St})}{\mathrm{d}c_{St}}\cdot\frac{\mathrm{d}c_{St}}{\mathrm{d}\eta}\right]\frac{\partial\Phi}{\mathrm{d}\eta}\\&\qquad+\left[\frac{\mathrm{d}D(c_{St})}{\mathrm{d}c_{St}}\cdot\frac{\mathrm{d}^2c_{St}}{\mathrm{d}\eta^2}+\frac{\mathrm{d}^2D(c_{St})}{\mathrm{d}c_{St}^2}\left(\frac{\mathrm{d}c_{St}}{\mathrm{d}\eta}\right)^2\right]\Phi.\end{aligned} \tag{8}$$

Equation (8) can be solved by the method of separation of variables:

$$\Phi(\phi,\eta)=f(\phi)g(\eta) \tag{9}$$

and since ϕ and η are independent variables we have,

$$\frac{4\dfrac{\mathrm{d}f}{\mathrm{d}\phi}(1-\phi v_0)}{f}=k=\text{a constant} \tag{10}$$

and

$$\begin{aligned}&\frac{D(c_{St})\dfrac{\mathrm{d}^2g}{\mathrm{d}\eta^2}}{g}+\frac{\left[2\eta v_0+2\dfrac{\mathrm{d}D}{\mathrm{d}c_{St}}\cdot\dfrac{\mathrm{d}c_{St}}{\mathrm{d}\eta}\right]\dfrac{\mathrm{d}g}{\mathrm{d}\eta}}{g}\\&\qquad+\left[\frac{\mathrm{d}D}{\mathrm{d}c_{St}}\cdot\frac{\mathrm{d}^2c_{St}}{\mathrm{d}\eta^2}+\frac{\mathrm{d}^2D}{\mathrm{d}c_{St}^2}\left(\frac{\mathrm{d}c_{St}}{\mathrm{d}\eta}\right)^2\right]=k.\end{aligned} \tag{11}$$

A close-form solution of the above equation being not readily available and since the primary objective is to test for stability conditions of the diffusion equation with a second-order concentration-dependent diffusivity, the following analysis is restricted to regions in the close vicinity of the interface, i.e. for small x.

Thus

$$\lim_{\eta\to 0}c_{St}(\eta)=\bar{c} \tag{12a}$$

$$\lim_{\eta\to 0}\frac{\mathrm{d}c_{St}}{\mathrm{d}\eta}(\eta)=-\frac{2}{\sqrt{\pi}}\bar{c} \tag{12b}$$

$$\lim_{\eta\to 0}\frac{\mathrm{d}^2c_{St}}{\mathrm{d}\eta^2}(\eta)=0 \tag{12c}$$

where

$$c_{St}\simeq\bar{c}\,\mathrm{erfc}\,\eta.$$

Taking the limit as $\eta\to 0$ in equations (11) and (12c) we have

$$D(\bar{c})\frac{\mathrm{d}^2g}{\mathrm{d}\eta^2}-\frac{4}{\sqrt{\pi}}\cdot\frac{\mathrm{d}D}{\mathrm{d}\bar{c}}\bar{c}\frac{\mathrm{d}g}{\mathrm{d}\eta}+\left[\frac{4}{\pi}\frac{\mathrm{d}^2D}{\mathrm{d}\bar{c}^2}-k\right]g=0. \tag{13}$$

This is a second order differential equation with constant coefficients and one can seek a solution readily with the substitution

$$g(\eta)=\mathrm{e}^{q\eta} \tag{14}$$

in equation (13).

Hence,

$$D(\bar{c})q^2-\frac{4}{\sqrt{\pi}}\frac{\mathrm{d}D}{\mathrm{d}\bar{c}}\bar{c}q+\left[\frac{4}{\pi}\frac{\mathrm{d}^2D}{\mathrm{d}\bar{c}^2}-k\right]=0$$

or

$$\begin{aligned}q=&+\frac{2}{\sqrt{\pi}\cdot D(\bar{c})}\cdot\frac{\mathrm{d}D}{\mathrm{d}\bar{c}}\bar{c}\\&\pm\left[\frac{16}{\pi}\left(\frac{\mathrm{d}D}{\mathrm{d}\bar{c}}\right)^2\bar{c}^2-4D(\bar{c})\left(\frac{4}{\pi}\frac{\mathrm{d}^2D}{\mathrm{d}\bar{c}^2}-k\right)\right]^{\frac{1}{2}}\cdot[2D(\bar{c})]^{-1}.\end{aligned} \tag{15}$$

The general solution is

$$g(\eta)\doteq A_1\,\mathrm{e}^{q_1\eta}+A_2\,\mathrm{e}^{q_2\eta} \tag{16}$$

with the boundary conditions:

$$g(0)=0\quad\text{and}\quad g(\infty)=0$$

or

$$g(\eta)=A_1(\mathrm{e}^{q_1\eta}-\mathrm{e}^{q_2\eta}). \tag{16a}$$

In order to satisfy the B.C. $g(\infty)=0$, it is necessary that q_1 and $q_2<0$, that is

$$\frac{\mathrm{d}D}{\mathrm{d}\bar{c}}<0. \tag{17}$$

Furthermore

$$\frac{16}{\pi}\left(\frac{\mathrm{d}D}{\mathrm{d}\bar{c}}\right)^2\bar{c}^2>\frac{16}{\pi}\left(\frac{\mathrm{d}D}{\mathrm{d}\bar{c}}\right)^2\bar{c}^2-4D\bar{c}\left[\frac{4}{\pi}\frac{\mathrm{d}^2D}{\mathrm{d}\bar{c}^2}-k\right]$$

or

$$4D(\bar{c})\left[\frac{4}{\pi}\frac{\mathrm{d}^2D}{\mathrm{d}\bar{c}^2}-k\right]>0,$$

that is,

$$\frac{4}{\pi}\frac{\mathrm{d}^2D}{\mathrm{d}\bar{c}^2}>k. \tag{18}$$

Now, equation (10) can be solved using the inequalities (17) and (18) to give:

$$\ln\left(\frac{f}{f_0}\right)=\int_0^{\phi}\frac{k}{4(1-\phi' v_0)}\mathrm{d}\phi'=-\frac{k}{4v_0}\ln(1-\phi v_0)$$

or

$$f(\phi)=f_0\cdot\frac{1}{(1-\phi v_0)+\dfrac{k}{4v_0}}=f_0(1-\phi v_0)^{-k/4v_0} \tag{19}$$

It can be seen readily that the perturbation at $t=0$ (or $\phi=0$) in equation (19) decays for $\phi=0\to\phi=1/v_0$. For $k\geqslant 0$, $f(\phi)\to\infty$. In other words we have an instability.

As k has to satisfy the inequality (18) instability is possible only when

$$0\leqslant k\frac{4}{\pi}\frac{\mathrm{d}^2D}{\mathrm{d}\bar{c}^2}. \tag{20}$$

That is, only if $\mathrm{d}^2D/\mathrm{d}\bar{c}^2$ is *positive*.

Substitution of equation (20) into equation (15) gives a condition for instability:

$$\frac{4}{(\sqrt{\pi})D(\bar{c})}\cdot\frac{\mathrm{d}D}{\mathrm{d}\bar{c}}\cdot\bar{c} < \mathrm{Real}(q) < 0. \tag{21}$$

It is evident that the more negative the term $\mathrm{d}D/\mathrm{d}\bar{c}$ is the larger the domain of q where instabilities occur. In the case of a constant diffusion coefficient, inequalities (20) and (21) reveal that no instability can occur.

The above analysis does not confirm the hypothesis that a minimum in the D–c relationship is necessary to induce instability but since such a configuration will contain a negative $\mathrm{d}D/\mathrm{d}c$ slope, such systems will give instabilities for concentrations up to the minimum value. Furthermore the analysis explains why some systems exhibit instability under conditions of absorption but not desorption and vice-versa.

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THE DISTURBANCE OF A UNIFORM STEADY-STATE HEAT FLUX BY A PARTIALLY CONDUCTING PLANE CRACK

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NOMENCLATURE

x, y, z,	cartesian co-ordinates;
r, θ, z,	cylindrical polar co-ordinates;
ρ, ζ,	non-dimensional polar co-ordinates;
$A, \bar{A}$,	extent of the crack and remainder of the crack plane respectively;
a,	crack radius or semi-width;
δ_{ij},	Kronecker delta;
h,	thermal contact resistance;
$j(t)$,	odd function of t defined by equation (8);
K,	thermal conductivity;
ϕ,	arccos t;
q_z,	heat flux density in z direction;
s, t,	parameters of integration;
T,	temperature.

INTRODUCTION

IF A THERMALLY conducting solid contains a small plane crack, the temperature field in the vicinity of the crack will be perturbed from that in an otherwise similar unflawed solid. The extreme case of a completely insulated crack leads to a classical mixed boundary value problem in potential theory, a solution for the penny-shaped crack being given by Karush and Young [1]. However, a more realistic boundary condition is that of "radiation" across the crack, proportional to the local discontinuity in temperature. This leads to a mixed boundary value problem of the third type which is here solved for the penny-shaped crack and the "Griffith" crack, using a technique developed in another context [2].

The same solution applies to the problem of a cooled semi-infinite solid, part of whose surface is obstructed.

It is hoped subsequently to use these results to find the thermal stresses in a solid containing a partially conducting crack.

STATEMENT OF THE PROBLEM

We define a system of cartesian co-ordinates (x, y, z) and polar co-ordinates (r, θ, z) such that the crack lies in the plane $z = 0$, denoting the extent of the crack by A and the rest of this plane by $\bar{A}$.

We assume that there is a uniform heat flux, q_0, in the z direction at the extremities of the solid, i.e.

$$q_z = -K\frac{\partial T}{\partial z} \rightarrow q_0, \quad z \rightarrow \pm\infty, \tag{1}$$

where T is the temperature and K the conductivity of the material.

In view of the antisymmetry of the problem there is no loss in generality in taking

$$T = 0, \quad \text{on} \quad \bar{A}, \tag{2}$$

in which case the local temperatures on opposite sides of the crack must be equal and opposite.

If the heat flux through the crack is proportional to the local temperature difference across it, we have

$$q_z = -2hT \quad \text{on} \quad A, \quad z = 0^+, \tag{3}$$

where h is the constant of proportionality.

Denoting the perturbation in temperature field due to the crack by T_1, such that

$$T = T_1 - \frac{q_0 z}{K} \tag{4}$$

and substituting for T into equation (1)–(3), we find

$$\frac{\partial T_1}{\partial z} \rightarrow 0, \quad z \rightarrow \infty, \tag{5}$$

$$T_1 = 0, \quad \text{on} \quad \bar{A}, \tag{6}$$

$$\frac{\partial T_1}{\partial z} - \frac{2hT_1}{K} = \frac{q_0}{K}, \quad \text{on} \quad A, z = 0^+. \tag{7}$$